II. Combining events: Multiplication and addition

Unions of events

- $A \cup B$ is the set of outcomes in A or B, meaning at least one of A or B is true.
- This is the inclusive OR: It means one or the other or both. (The exclusive OR, meaning one or the other but not both, is not as common.)

Think of soup or salad at a restaurant. This isn't that.

Draw a Venn diagram with S, A, and B.

- $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$
- If A and B are disjoint events (no overlap), then $P(A \cap B) = 0$, so we get

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

Draw a Venn diagram with S, A, and B.

Intersections of events

- $A \cap B$ is the set of outcomes in A and B, meaning both of A and B are true.

Draw a Venn diagram with S, A, and B.

- To find $P(A \cap B)$, the multiplication rule is often useful:
- If you have m possible outcomes for one experiment and n possible outcomes for a second, independent experiment, then there are mn possible combined outcomes.

Conditional probability

- The conditional probability $P(A|B)$ ("$P(A$ given $B)$") is the probability of event A, given that event B is true.

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

Draw a Venn diagram with S, A, and B.

Independence

- Events A and B are independent if knowing that B is true would not change the probability that A is true:

$$\boxed{P(A) = P(A|B)}$$

- **Warning**: Independent does not mean disjoint. (Disjoint means that $P(A \cap B) = 0$.)
- If A and B are independent, then $P(A) = \frac{P(A \cap B)}{P(B)}$, so we get

$$\boxed{P(A \cap B) = P(A)P(B)}.$$

Example I

Choose a whole number at random from 1 to 100. What is the probability that it will be even or greater than 80?

By counting:

$$\begin{aligned}
S &:= \{\text{all numbers}\} \\
A &:= \{\text{even numbers}\} \\
B &:= \{\text{numbers } > 80\} \\
|A| &= 50 \\
|B| &= 20 \\
|A \cap B| &= 10 \\
|A \cup B| &= |A| + |B| - |A \cap B| = 50 + 20 - 10 = 60 \\
P(A \cup B) &= \frac{|A \cup B|}{|S|} = \frac{60}{100} = \boxed{\frac{3}{5}}
\end{aligned}$$

By probability:

$$\begin{aligned}
P(A) &= \frac{50}{100} = \frac{1}{2} \\
P(B) &= \frac{20}{100} = \frac{1}{5} \\
P(A \cap B) &= \frac{10}{100} = \frac{1}{10} \\
P(A \cup B) &= P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{5 + 2 - 1}{10} = \boxed{\frac{3}{5}}
\end{aligned}$$

Example II

A combination deal at a restaurant includes a main dish, drink, and dessert. If the restaurant serves four types of drinks, 12 main dishes, and three desserts, then how many different combination meals are possible?

$4 \cdot 12 \cdot 3 = \boxed{144}$

Example III

We roll two dice, a red one and a blue one. What is the probability that the red one shows a 4 and the blue one is odd?

By counting:

$$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{|\{41, 43, 45\}|}{36} = \boxed{\frac{1}{12}}$$

By probability:
These are independent, since knowing that one is true would not change the the probability that the other is true.

$$P(A \cap B) = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \boxed{\frac{1}{12}}$$

Example IV

Flip a coin four times. What is the probability that there will be at least three heads, given that the first two flips are heads?

$$\begin{aligned}
S &:= \{\text{all 16 outcomes}\} \\
A &:= \{\text{at least three heads: HHHT, HHTH, HTHH, THHH, HHHH}\} \\
B &:= \{\text{first two are heads: HHHH, HHHT, HHTH, HHTT}\} \\
A \cap B &= \{HHHT, HHTH, HHHH\} \\
P(A|B) &= \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{16}}{\frac{4}{16}} = \boxed{\frac{3}{4}}
\end{aligned}$$

Example V

Roll two dice (red and blue). Define the events:

$$\begin{aligned} A &= \text{red die is 3} \\ B &= \text{total is 7} \\ C &= \text{total is 8} \end{aligned}$$

Calculate the conditional probabilities:

$$\begin{aligned} &P(B|A) \\ &P(A|B) \\ &P(C|A) \\ &P(A|C) \\ &P(C|B) \\ &P(B|C) \end{aligned}$$

$A =$ red die is 3. $P(A) = \frac{6}{36} = \frac{1}{6}$.
$B =$ total is 7. $P(B) = \frac{6}{36} = \frac{1}{6}$.
$C =$ total is 8. $P(C) = \frac{5}{36}$.

$$P(B|A) = \boxed{\frac{1}{6}}$$
$$P(A|B) = \boxed{\frac{1}{6}}$$
$$P(C|A) = \boxed{\frac{1}{6}}$$
$$P(A|C) = \boxed{\frac{1}{5}} \quad \text{Different from } P(C|A)!$$
$$P(C|B) = \boxed{0}$$
$$P(B|C) = \boxed{0}$$

Example VI

As above, roll two dice (red and blue). Define the events:

$A =$ red die is 3
$B =$ total is 7
$C =$ total is 8

Are events A and B independent? A and C? B and C?

By the formulas, A and B are [independent]. A and C are [dependent]. B and C are [dependent].