XII. Binomial Distribution (Bernoulli Trials)

Binomial Distribution

- The binomial distribution describes a sequence of n independent tests, each of which can have two outcomes.

Success or failure.

- These are also known as Bernoulli trials.
- The prototypical example is flipping a coin n times.
- But it can also describe any process with two outcomes (games between teams, rolling a die to get a 6, etc.).

Formula for the Binomial Distribution

- Fixed parameters:

$$\begin{aligned} n &:= \text{number of trials} \\ p &:= \text{probability of success on any given trial} \\ q &:= \text{probability of failure } = 1 - p \end{aligned}$$

- Random variable:

$$Y := \text{number of successes (heads)}$$

- Probability distribution:

$$\boxed{p(y) = \binom{n}{y} p^y q^{n-y}, 0 \le y \le n}$$

$\binom{n}{y}$ is not a fraction.

$$\binom{n}{y} = C_y^n = \frac{n!}{y!(n-y)!}$$

Warning: These are different p's!
p is the probability of success on any given trial.
$p(y)$ is the probability of y successes overall.

Key Properties of the Binomial Distribution

- **Mean**:

$$\boxed{\mu = E(Y) = np}$$

- **Variance**:

$$\boxed{\sigma^2 = V(Y) = npq}$$

- **Standard deviation**:

$$\boxed{\sigma = \sqrt{V(Y)} = \sqrt{npq}}$$

Example I

The Los Angeles Angels play the Tasmania Devils in a five-game series. If the Angels have a $\frac{1}{3}$ chance of winning any given game, what is the chance that they will win exactly three games?

$$\begin{aligned} p(y) &= \binom{n}{y} p^y q^{n-y}, 0 \le y \le n \\ \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 &= \frac{10 \cdot 4}{3^5} = \frac{40}{243} \approx \boxed{\frac{1}{6}} \end{aligned}$$

Example II

You study enough for an exam to have a $\frac{3}{4}$ chance of solving any given problem. The exam is 10 problems. What is your chance of scoring 80% or better?

$$\begin{aligned} p(y) &= \binom{n}{y} p^y q^{n-y}, 0 \le y \le n \\ p = \frac{3}{4}, n = 10 \implies & \binom{10}{8}\left(\frac{3}{4}\right)^8\left(\frac{1}{4}\right)^2 + \binom{10}{9}\left(\frac{3}{4}\right)^9\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^{10} \\ &\approx 0.2816 + 0.1877 + 0.0563 \\ &= 0.5256 \\ &\approx \boxed{53\%} \end{aligned}$$

Example III

On the exam from Example II, each problem is worth 10 points. What is your expected grade? What is your standard deviation?

$$E(Y) = np = 10 \cdot \frac{3}{4} \text{ problems } = \boxed{75 \text{ points}}$$

Your expected grade is 75. Actually you will get 60, 70, 80, 90, or 100, but your average over many exams will be 75.

$$\begin{aligned} V(Y) &= npq = 10 \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{30}{16} \\ \sigma &= \sqrt{\frac{30}{16}} = \frac{\sqrt{30}}{4} \approx 1.369 \text{ problems } \approx \boxed{14 \text{ points}} \end{aligned}$$

Example IV

Each year, the Long Beach Jackrabbits have an 80% chance of winning the world pogo-sticking championship.

A. What is the probability that Jackrabbits will win exactly five times in the next seven years?

B. What is the probability that Jackrabbits will win at least five times in the next seven years?

Example IV

A. Exactly five times in seven years?

B. At least five times in seven years?

A.

$$p(5) = \binom{7}{5}\left(\frac{4}{5}\right)^5\left(\frac{1}{5}\right)^2 = \frac{7\cdot 6}{2}\frac{2^{12}}{5^7} = \boxed{\frac{21\cdot 4^5}{5^7}}$$

B.

$$\begin{aligned} p(5)+p(6)+p(7) &= \frac{21\cdot 4^5}{5^7} + \binom{7}{6}\left(\frac{4}{5}\right)^6\frac{1}{5} + \binom{7}{7}\left(\frac{4}{5}\right)^7 \\ &= \frac{21\cdot 4^5 + 7\cdot 4^6 + 4^7}{5^7} \\ &= \frac{4^5(21+28+16)}{5^7} \\ &= \frac{4^5\cdot 65}{5^7} \\ &= \boxed{\frac{4^5\cdot 13}{5^6}} \end{aligned}$$

Example V

Each year, the Long Beach Jackrabbits have an 80% chance of winning the world pogo-sticking championship. Find the expected number of championships the Jackrabbits will win in the next five years and the standard deviation.

$$E(Y) = np = 5 \cdot \frac{4}{5} = \boxed{4 \text{ championships}}$$

$$\begin{aligned} V(Y) &= npq = 5 \cdot \frac{4}{5} \cdot \frac{1}{5} = \frac{4}{5} \\ \sigma &= \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} \approx \boxed{0.894 \text{ championships}} \end{aligned}$$